

The characteristic polynomial of an algebra and representations

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Suppose that \mathbf{k} is a field and let A be a finite dimensional, associative, unital \mathbf{k} -algebra. Often one is interested in studying the finite-dimensional representations of A . Of course, a finite dimensional representation of A is simply a finite dimensional \mathbf{k} -vector space M and a \mathbf{k} -algebra homomorphism $A \rightarrow \text{End}_{\mathbf{k}}(M)$. In this article we will not consider representations of algebras, but rather how to determine if a \mathbf{k} -linear map $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ is actually a homomorphism. We restrict our attention to the case where A is a product of field extensions of \mathbf{k} . If $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ is a representation then certainly, if $a \in A$ satisfies $a^m = 1$ then $\phi(a)^m = \text{id}$ as well. Our first Theorem is a remarkable converse to this elementary observation.

Theorem A. *Suppose that $A = \mathbf{k}^d$ and $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ is a \mathbf{k} -linear map. Let $n > 2$ be a natural number and assume that \mathbf{k} has n primitive n^{th} roots of unity. If $\phi(1_A) = \text{id}$ and for each $a \in A$ such that $a^n = 1_A$ we have $\phi(a)^n = \text{id}$, then ϕ is an algebra homomorphism.*

Consider the regular representation $\mu_L : A \rightarrow \text{End}_{\mathbf{k}}(A)$ of A on itself by left multiplication. For $a \in A$, let $\chi_a(t)$ and $\bar{\chi}_a(t)$ be the characteristic and minimal polynomials of $\mu_L(a)$, respectively. We note that $\chi_a(a) = \bar{\chi}_a(a) = 0$ in A . Therefore if M is a finite dimensional left A module with structure map $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ then $\chi_a(\phi(a)) = \bar{\chi}_a(\phi(a)) = 0$ in $\text{End}_{\mathbf{k}}(M)$. The notion of assigning a characteristic polynomial to each element of an algebra and considering representations which are compatible with this assignment has appeared in [Pro87]. This idea has been applied to some problems in noncommutative geometry as well [LB03]. However, as far as we know the following related notion is new.

Definition 1. Suppose that $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ is a \mathbf{k} -linear map, where M is a finite dimensional \mathbf{k} -vector space. We say that ϕ is a *characteristic morphism* if $\chi_a(\phi(a)) = 0$ for all $a \in A$. We say that ϕ is *minimal-characteristic* if, moreover, $\bar{\chi}_a(\phi(a)) = 0$ for all $a \in A$.

It is natural to ask whether or not the notions of characteristic morphism and minimal characteristic morphism are weaker than the notion of algebra morphism. Let us address minimal-characteristic morphisms first.

Corollary. *Assume that $A = \mathbf{k}^d$ and that \mathbf{k} has a full set of d^{th} roots of unity. Then a minimal-characteristic morphism $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ is an algebra morphism.*

Proof. First note that $\bar{\chi}_1(t) = t - 1$. Hence $\phi(1) = \text{id}$. Furthermore if $a \in A$ satisfies $a^d = 1$ then $\bar{\chi}_a(t)$ divides $t^d - 1$. Therefore, $\phi(a)^d = \text{id}$. Hence, if $d > 2$ then Theorem A implies that ϕ is an algebra morphism. We leave the cases $d = 1, 2$ for the reader. ■

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Example 2. Let $a, b \in \mathbf{k}$ be such that $a + b \neq 0$. Then the map $\phi : \mathbf{k}^{\times 2} \rightarrow \text{Mat}_2(\mathbf{k})$ given by

$$\phi(e_1) = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \quad \phi(e_2) = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

is a characteristic morphism that is not a representation.

Characteristic morphisms form a category in a natural way. Any linear map $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ endows M with the structure of a $T(A)$ module, where $T(A)$ denotes the tensor algebra on A . We can view the characteristic polynomial of elements of A as a homogeneous form $\chi(t) \in \text{Sym}^\bullet(A^\vee)[t]$ of degree d , monic in t . Pappacena [Pap00] associates to such a form an algebra

$$C(A) = \frac{T(A)}{\langle \chi_a(a) : a \in A \rangle},$$

where if $\chi_a(t) = \sum_{i=0}^d c_i(a)t^i$ then

$$\chi_a(a) := \sum_{i=0}^d c_i(a)a^{\otimes i} \in T(A).$$

Clearly, the action map $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ of a $T(A)$ -module M is a characteristic morphism if and only if the action of $T(A)$ factors through $C(A)$. We declare the category of characteristic morphisms to be the category of finite-dimensional $C(A)$ -modules. So we have a notion of irreducible characteristic morphism. The characteristic morphism constructed in Example 2 is not irreducible, being an extension of two irreducible characteristic morphisms. However, every irreducible characteristic morphism $\mathbf{k}^{\times 2} \rightarrow \text{End}_{\mathbf{k}}(M)$ is actually an algebra morphism. The following example shows this is not always the case.

Example 3. The linear map $\phi : \mathbf{k}^{\times 3} \rightarrow \text{Mat}_3(\mathbf{k})$ defined by

$$e_1 \mapsto \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad e_2 \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad e_3 \mapsto \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

is an irreducible characteristic morphism, but not an algebra morphism; while $e_1^2 = e_1$, it can be checked that $\phi(e_1)^2 \neq \phi(e_1)$.

Given a linear map $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$, let $T_\phi \in A^\vee \otimes \text{End}_{\mathbf{k}}(M)$ be the element that corresponds to ϕ under the isomorphism $\text{Hom}_{\mathbf{k}}(A, \text{End}_{\mathbf{k}}(M)) \cong A^\vee \otimes \text{End}_{\mathbf{k}}(M)$. We view T_ϕ as an element of $\text{Sym}^\bullet(A^\vee) \otimes \text{End}_{\mathbf{k}}(M)$. The equation $\chi_a(\phi(a)) = 0$ for all $a \in A$ holds if and only if $\chi_A(T_\phi) = 0$ in $\text{Sym}^\bullet(A^\vee) \otimes \text{End}_{\mathbf{k}}(M)$. We can just as easily view T_ϕ as an element of $T(A^\vee) \otimes \text{End}_{\mathbf{k}}(M)$. Moreover, we can lift χ from $\text{Sym}^\bullet(A^\vee)[t]$ to $T(A^\vee) * \mathbf{k}[t]$ by the naïve symmetrization map $\text{Sym}^\bullet(A^\vee)[t] \rightarrow T(A^\vee) * \mathbf{k}[t]$.

Theorem B. Assume that $\text{char}(\mathbf{k})$ is either 0 or greater than d . Let $A = \mathbf{k}^{\times d}$ and $\phi : A \rightarrow \text{End}_{\mathbf{k}}(M)$ a \mathbf{k} -linear map. The map ϕ factors through a homomorphism $A \rightarrow \text{End}_{\mathbf{k}}(M)$ if and only if $\chi(T) = 0$ in $T(A^\vee) \otimes_{\mathbf{k}} B$.

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Proofs

We now turn to the proofs of the results in the introduction. The proof of the Theorem A depends on an arithmetic Lemma.

Lemma 4. *Let $\zeta \in \mathbf{k}$ be a primitive n^{th} root of unity. Suppose that $a, b, c, d \in \mathbb{Z}$ satisfy $b, d \not\equiv 0 \pmod n$ and*

$$\frac{\zeta^a - 1}{\zeta^b - 1} = \frac{\zeta^c - 1}{\zeta^d - 1}.$$

Then either:

1. $a \equiv b \pmod n$ and $c \equiv d \pmod n$, or
2. $a \equiv c \pmod n$ and $b \equiv d \pmod n$.

Proof. After possibly passing to a finite extension we may assume that \mathbf{k} admits an automorphism sending ζ to ζ^{-1} . Thus we have

$$\frac{\zeta^{-a} - 1}{\zeta^{-b} - 1} = \frac{\zeta^{-c} - 1}{\zeta^{-d} - 1},$$

which we rewrite

$$\frac{\zeta^{-a}}{\zeta^{-b}} \cdot \frac{1 - \zeta^a}{1 - \zeta^b} = \frac{\zeta^{-c}}{\zeta^{-d}} \cdot \frac{1 - \zeta^c}{1 - \zeta^d}.$$

Using our assumption we find that $\zeta^{b-a} = \zeta^{d-c}$. Thus $b - a \equiv d - c \pmod n$. Let $e = b - a \equiv d - c \pmod n$. Then we have

$$\frac{\zeta^{b-e} - 1}{\zeta^b - 1} = \frac{\zeta^{d-e} - 1}{\zeta^d - 1}$$

which implies that

$$\zeta^{b-e} + \zeta^d = \zeta^{d-e} + \zeta^b.$$

Finally we see that

$$\zeta^d - \zeta^b = (\zeta^d - \zeta^b)\zeta^{-e}$$

Therefore either $e \equiv 0 \pmod d$ so that (1) holds, or $d \equiv b$ so that (2) holds. ■

Proof of Theorem A. Whether or not ϕ is an algebra homomorphism is stable under passage to the algebraic closure of \mathbf{k} . So we may assume that \mathbf{k} is algebraically closed. Let $e_1, \dots, e_d \in A$ be a complete set of orthogonal idempotents. Put $\alpha_i = \phi(e_i)$ and note that by hypothesis $\alpha_1 + \dots + \alpha_d = \text{id}$. Fix a primitive n^{th} root of unity ξ . Then $x = 1 + (\xi - 1)e_i$ satisfies $x^n = 1$. Therefore $\phi(x)^d = \text{id}$. This implies that $\phi(x)$ is diagonalizable and each eigenvalue is an n^{th} root of unity. Now, since ϕ is linear,

$$\alpha_i = \frac{\phi(x) - \text{id}}{\xi - 1}$$

and hence α_i is diagonalizable as well. Let λ be an eigenvalue of α_i . Then for some a we have

$$\lambda = \frac{\xi^a - 1}{\xi - 1}.$$

Now for any b , $\phi(1 + (\xi^b - 1)e_i)^d = \text{id}$. So we see that

$$1 + \lambda(\xi^b - 1)$$

must be a root of unity for every b . However, if

$$1 + \lambda(\xi^b - 1) = \xi^c$$

then Lemma 4 implies that either $a \equiv 1 \pmod{n}$, $\lambda = 0$, or $b \equiv 1 \pmod{n}$. Now, b is under our control and since $n \geq 3$ we can choose $b \not\equiv 0, 1 \pmod{n}$, excluding the third case. If $a \equiv 1 \pmod{n}$ then $\lambda = 1$ and otherwise $\lambda = 0$. Thus α_i is semisimple with eigenvalues equal to zero or one. So $\alpha_i^2 = \alpha_i$.

Let $i \neq j$ and consider

$$y_a = \text{id} + (\xi^a - 1)(\alpha_i + \alpha_j)$$

Clearly, $y_a^n = \text{id}$ and thus y_a is semisimple. We compute

$$(y_a - \text{id})^2 = (\xi^a - 1)^2(\alpha_i\alpha_j + \alpha_j\alpha_i) + (\xi^a - 1)(y_a - \text{id})$$

and deduce that

$$(\xi_a - 1)^{-2}(y_a - \text{id})(y_a - \xi^a) = (\alpha_i\alpha_j + \alpha_j\alpha_i). \quad (1)$$

Assume that $b \not\equiv 0 \pmod{n}$. Observe that $y_a - \text{id} = \frac{\xi^a - 1}{\xi^b - 1}(y_b - \text{id})$ and therefore, y_a and y_b are simultaneously diagonalizable. Suppose that ξ^c is an eigenvalue of y_b unequal to 1. Then

$$\frac{\xi^a - 1}{\xi^b - 1}(\xi^c - 1) + 1 = \xi^e$$

is an eigenvalue of y_a . Since $n \geq 3$ we can assume that $a \not\equiv b, 0 \pmod{n}$. Then Lemma 4 implies that $e \equiv a \pmod{n}$ and $b \equiv c \pmod{n}$. We see that the only eigenvalues of y_b are 1 and ξ^b .

Because y_b is semisimple, this means that the right side of (1) vanishes. So $\alpha_i\alpha_j = -\alpha_j\alpha_i$ for all i, j . Suppose that $\alpha_i(m) = m$. Then $\alpha_j(\alpha_i(m)) = \alpha_j(m) = -\alpha_i(\alpha_j(m))$. Since -1 is not an eigenvalue of α_i we see that $\alpha_j(m) = 0$. Now let $m \in M$ and write $m = m_0 + m_1$ where $\alpha_i(m_0) = 0$ and $\alpha_i(m_1) = m_1$. Then

$$\alpha_i(\alpha_j(m)) = \alpha_i(\alpha_j(m_0)) = -\alpha_j(\alpha_i(m_0)) = 0.$$

Thus we see that in fact $\alpha_i\alpha_j = 0$. So $\alpha_1, \dots, \alpha_d$ satisfy the defining relations of $\mathbf{k}^{\times d}$ and ϕ is actually an algebra homomorphism. \blacksquare

We now turn to the proof of Theorem B. The key idea is to use the fact that the single equation $\chi(T_\phi) = 0$ over the tensor algebra encodes many relations for the matrices defining ϕ . It is convenient to consider $\alpha_i = \phi(e_i)$, where e_i is the standard basis of idempotents in $\mathbf{k}^{\times d}$. Furthermore we write χ_d for the characteristic polynomial of $\mathbf{k}^{\times d}$ viewed as an element of $\mathbf{k}\langle x_1, \dots, x_d, t \rangle$ (where x_1, \dots, x_d is the dual basis to e_1, \dots, e_d).

Lemma 5. *Suppose that \mathbf{k} is a field with $\text{char}(\mathbf{k}) > d$. Let $\alpha_1, \dots, \alpha_d \in M_n(\mathbf{k})$ and put $T = x_1\alpha_1 + \dots + x_d\alpha_d$. If T satisfies χ_d then*

1. *for some $i = 1, \dots, d$, α_i has a 1-eigenvector, and*
2. *if $m \in \mathbf{k}^n$ satisfies $\alpha_i m = m$ then $\alpha_j m = 0$ for all $j \neq i$.*

Proof. (1.) Let $S = \mathbf{k}[x_1, \dots, x_d]$ as an $A = \mathbf{k}\langle x_1, \dots, x_d \rangle$ module in the obvious way. Then the image of χ_d in $\mathbf{k}[x_1, \dots, x_d, t]$ is $p(t) = n!(t - x_1) \cdots (t - x_d)$, where now the order of the terms does not matter. Hence T satisfies $(T - x_1) \cdots (T - x_d) = 0$ in $M_n(S)$. So we can view S^n as an $R = \mathbf{k}[x_1, \dots, x_d, t]/(p(t))$ -module M . For each i consider the quotient $S_i := R/(t - x_i)$, which is isomorphic to S under the natural map $S \rightarrow S_i$. Define $M_i = M \otimes_R S_i$. Since the map $S \rightarrow S_1 \times \cdots \times$

S_d is an isomorphism after inverting $a = \prod_{i \neq j} (x_i - x_j)$ and a is a nonzerodivisor on M , the natural map $M \rightarrow M_1 \oplus \dots \oplus M_d$ is injective. Hence there is some i such that M_i has positive rank. Consider $\bar{M} := M/(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_d)M$ and $\bar{M}_i := M_i/(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_d)M_i$. Now since M_i (is finitely generated and) has positive rank $\bar{M}_i \neq 0$. Observe that since $M = S^d$, the natural map $\mathbf{k}^d \rightarrow \bar{M}$ is an isomorphism. Moreover the action of t on \bar{M} is identified with the action of α_i . Now, $\bar{M}_i = \bar{M}/(t - x_i)\bar{M} = \bar{M}/(\alpha_i - 1)\bar{M} \neq 0$. Hence $\alpha_i - 1$ is not invertible, $\alpha_i - 1$ has nonzero kernel, and α_i has a 1-eigenvector.

(2.) Let us compute $\chi(x_1, \dots, x_d, T)$. We denote by δ_j^i the Kronecker function. We have

$$\begin{aligned} \chi_d(x_1, \dots, x_d, T) &= \sum_{\sigma \in S_d} \left(\sum_{i=1}^d x_i \alpha_i - x_{\sigma(1)} \right) \cdots \left(\sum_{i=1}^d x_i \alpha_i - x_{\sigma(d)} \right) \\ &= \sum_{\sigma \in S_d} \prod_{j=1}^d \left(\sum_{i=1}^d x_i (\alpha_i - \delta_{\sigma(j)}^i) \right) \\ &= \sum_{1 \leq i_1, \dots, i_d \leq d} x_{i_1} \cdots x_{i_d} \left(\sum_{\sigma \in S_d} (\alpha_{i_1} - \delta_{\sigma(1)}^{i_1}) \cdots (\alpha_{i_d} - \delta_{\sigma(d)}^{i_d}) \right). \end{aligned}$$

In the second line the term order matters so the product is taken in the natural order $j = 1, 2, \dots, d$. Now suppose that $\chi_d(x_1, \dots, x_d, T) = 0$. Then for all $1 \leq i_1, \dots, i_d \leq d$ we have

$$\sum_{\sigma \in S_d} (\alpha_{i_1} - \delta_{\sigma(1)}^{i_1}) \cdots (\alpha_{i_d} - \delta_{\sigma(d)}^{i_d}) = 0. \quad (2)$$

For each $j \neq i$, we consider the noncommutative monomial $x_i x_j x_i^{d-2}$ and its equation (2),

$$\sum_{\sigma \in S_d} (\alpha_i - \delta_{\sigma(1)}^i)(\alpha_j - \delta_{\sigma(2)}^j)(\alpha_i - \delta_{\sigma(3)}^i) \cdots (\alpha_i - \delta_{\sigma(d)}^i) = 0. \quad (3)$$

Note that since $\alpha_i(m) = m$, we calculate

$$(\alpha_i - \delta_{\sigma(3)}^i) \cdots (\alpha_i - \delta_{\sigma(d)}^i) m = \begin{cases} m & i \notin \{\sigma(3), \dots, \sigma(d)\}, \\ 0 & i \in \{\sigma(3), \dots, \sigma(d)\}. \end{cases}$$

Therefore applying (3) to m and simplifying we get

$$\begin{aligned} \sum_{\sigma \in S_d, \sigma(1)=i} (\alpha_i - 1)(\alpha_j - \delta_{\sigma(2)}^j) m + \sum_{\sigma \in S_d, \sigma(2)=i} \alpha_i \alpha_j m &= (d-1)!((\alpha_i - 1)(\alpha_j - \delta_{\sigma(2)}^j) m + \alpha_i \alpha_j m) \\ &= (d-1)!((\alpha_i - 1)\alpha_j m + \alpha_i \alpha_j m) \\ &= (d-1)!(2\alpha_i - 1)\alpha_j m \\ &= 0, \end{aligned}$$

where passing from the first line to the second we use the fact that $(\alpha_i - 1)\delta_{\sigma(2)}^j m = 0$.

Now, consider the special case of (2) corresponding to x_i^d :

$$\sum_{\sigma \in S_d} (\alpha_i - \delta_{\sigma(1)}^i) \cdots (\alpha_i - \delta_{\sigma(d)}^i) = \sum_{j=1}^d \sum_{\sigma \in S_d, \sigma(j)=i} \alpha_i^{j-1} (\alpha_i - 1) \alpha_i^{d-j-1} = d! \alpha_i^{d-1} (\alpha_i - 1) = 0.$$

Since $\alpha_i^{d-1}(\alpha_i - 1) = 0$ it follows that $2\alpha_i - 1$ is invertible. However, $(2\alpha_i - 1)\alpha_j m = 0$ so $\alpha_j m = 0$. ■

Proof of Theorem B. (\Leftarrow) We proceed by induction on $\dim(M)$ and fix an identification $M \cong \mathbf{k}^n$. Suppose $n = 1$. Then by Lemma 5, there is some i and some $m \in \mathbf{k}^1$ such that $\alpha_i(m) = m$. Moreover, $\alpha_j m = 0$ for all $j \neq 0$. Since m spans \mathbf{k}^1 , the α_i satisfy the necessary relations for ϕ to factor through an algebra morphism.

Now given $\alpha_1, \dots, \alpha_d \in M_n(\mathbf{k})$, Lemma 5 implies that we can find an element $m \in \mathbf{k}^d$ such that $\mathbf{k}m \subset \mathbf{k}^d$ is stable under the action of $\alpha_1, \dots, \alpha_d$. Let $M_n(\mathbf{k}, m) \subset M_n(\mathbf{k})$ be the algebra of operators that preserve $\mathbf{k}m$. Then there is a surjective algebra homomorphism $M_n(\mathbf{k}, m) \twoheadrightarrow M_{n-1}(\mathbf{k})$. Since $\alpha_1, \dots, \alpha_d \in M_n(\mathbf{k}, m)$ we find that $T \in M_n(\mathbf{k}, m) \otimes_{\mathbf{k}} \mathbf{k}\langle x_1, \dots, x_d \rangle$. So if $\alpha'_1, \dots, \alpha'_d \in M_{n-1}(\mathbf{k})$ are the images of $\alpha_1, \dots, \alpha_d$ then $T' = x_1 \alpha'_1 + \dots + x_d \alpha'_d$ satisfies χ_d . By induction we see that $(\alpha'_i)^2 = \alpha'_i$ and $\alpha'_i \alpha'_j = 0$ for $i \neq j$. In particular, there is a codimension 1 subspace of \mathbf{k}^{n-1} preserved by $\alpha'_1, \dots, \alpha'_d$. Its inverse image in \mathbf{k}^n (we identify \mathbf{k}^{n-1} with $\mathbf{k}^n / \mathbf{k}m$) is then a codimension one subspace $V' \subset \mathbf{k}^d$ which is invariant under $\alpha_1, \dots, \alpha_d$. Again by induction, $\alpha_i^2 - \alpha_i$ and $\alpha_i \alpha_j$ ($i \neq j$) annihilate V' . There is some i such that α_i acts by the identity on \mathbf{k}^n / V' . Since $\alpha_i^{d-1}(\alpha_i - 1) = 0$, the geometric multiplicity of 1 as an eigenvalue of α_i is equal to its algebraic multiplicity. So there is a 1-eigenvector $m \in \mathbf{k}^n$ whose image in \mathbf{k}^n / V' is nonzero. Again Lemma 5 implies that $\alpha_j m = 0$ for $j \neq i$. Hence the relations $\alpha_i^2 - \alpha_i$ and $\alpha_i \alpha_j$ annihilate a basis for \mathbf{k}^n and hence annihilate \mathbf{k}^n .

(\Rightarrow) Suppose that ϕ is an algebra map. Then we have $\alpha_i^2 = \alpha_i$ for all i and $\alpha_j \alpha_i = 0$ if $i \neq j$. Decompose $\mathbf{k}^n = V_1 \oplus \dots \oplus V_d$ where $V_i = \alpha_i(\mathbf{k}^n)$. Then T preserves $V_i \otimes \mathbf{k}\langle x_1, \dots, x_d \rangle$ for each i . So we can view T as an element of $\prod_{i=1}^d \text{End}_{\mathbf{k}}(V_i) \otimes \mathbf{k}\langle x_1, \dots, x_d \rangle \subset M_n(\mathbf{k}\langle x_1, \dots, x_d \rangle)$. Since $(T - x_i)$ vanishes identically on $V_i \otimes \mathbf{k}\langle x_1, \dots, x_d \rangle$ we see that for each $\sigma \in S_d$ and each i the image of $(T - x_{\sigma(1)}) \cdots (T - x_{\sigma(d)})$ vanishes in $\text{End}_{\mathbf{k}}(V_i) \otimes \mathbf{k}\langle x_1, \dots, x_d \rangle$ and hence in $M_n(\mathbf{k}\langle x_1, \dots, x_d \rangle)$. Since all of the terms of $\chi_d(T)$ vanish in $M_n(\mathbf{k}\langle x_1, \dots, x_d \rangle)$, so does $\chi_d(T)$. \blacksquare

Questions

There are many natural questions that surround the notion of characteristic morphism. We point out a few of them.

Question 1. *What are the irreducible characteristic morphisms for $A = \mathbf{k}^{\times d}$? Are there infinitely many for $d \geq 3$?*

Replacing a commutative semisimple algebra with a semisimple algebra, Theorem A fails to hold. Indeed, the map $\phi : \text{Mat}_d(\mathbf{k}) \rightarrow \text{Mat}_d(\mathbf{k})$ defined by $\phi(M) = M^T$ is not a homomorphism, but does satisfy the hypotheses of Theorem A. Moreover, ϕ is a characteristic morphism.

Question 2. *Is there a characterization of when a linear map $\phi : \text{Mat}_d(\mathbf{k}) \rightarrow \text{Mat}_r(\mathbf{k})$ is a homomorphism along the lines of Theorem A?*

Let V is a finite dimensional vector space and $F(t) \in \text{Sym}^\bullet(V^\vee)[t]$ be monic and homogeneous. Given $v \in V$ we can consider the image $F_v(t)$ of $F(t)$ under the homomorphism $\text{Sym}^\bullet(V^\vee)[t] \rightarrow \mathbf{k}[t]$ induced by $v : V^\vee \rightarrow \mathbf{k}$. The main theorem of [CK15] implies that there always exists a linear map $\phi : V \rightarrow \text{Mat}_r(\mathbf{k})$ for some r such that $F_v(\phi(v)) = 0$ for all $v \in V$. There is a natural non-commutative generalization of this problem.

Question 3. *For which monic, homogenous elements $F(t)$ of $\text{T}(V^\vee) * \mathbf{k}[t]$, does there exist an element $\phi^\vee \in V^\vee \otimes \text{Mat}_r(V)$ for some r such that $F(\phi^\vee) = 0$ in $\text{T}(V^\vee) \otimes \text{Mat}_r(\mathbf{k})$?*

If $F(t)$ is the symmetrization of the characteristic polynomial of an algebra structure on V then we have an affirmative answer. However, if $F(t) = t^2 - u \otimes v$ where u, v are linearly independent, then there is no such element.

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